Exercise 27

Prove the statement using the precise definition of a limit.

$$\lim_{x\to 2}(x^2 - 3x) = -2$$

Solution

Proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-2| < \delta \qquad \text{then} \qquad |(x^2-3x)-(-2)| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-2|$.

$$|(x^2-3x)-(-2)| < \varepsilon$$

$$|x^2-3x+2| < \varepsilon$$

$$|(x-1)(x-2)| < \varepsilon$$

$$|x-1||x-2| < \varepsilon$$

On an interval centered at $x = 2$, a positive constant C can be chosen so that $|x-1| < C$.

$$C|x-2| < \varepsilon$$

$$|x-2| < \frac{\varepsilon}{C}$$

To determine C, suppose that x is within a distance a from 2.

$$|x-2| < a$$

$$-a < x-2 < a$$

$$-a+1 < x-1 < a+1$$

$$|x-1| < a+1$$

The constant C is then $a+1$. Choose δ to be whichever is smaller between a and $\varepsilon/(a+1)$: $\delta = \min\{a, \varepsilon/(a+1)\}$. Now, assuming that $|x-2| < \delta$,

$$\begin{aligned} |(x^2-3x)-(-2)| &= |x^2-3x+2| \\ &= |(x-1)(x-2)| \\ &= |x-1||x-2| \\ &< C\delta = (a+1)\left(\frac{\varepsilon}{a+1}\right) = \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 2}(x^2 - 3x) = -2.$$